On the Marriage of Chiral Perturbation Theory and Dispersion Relations

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Abstract

I describe the methodology for the use of dispersion relations in connection with chiral perturbation theory. The conditions for matching the two formalisms are given at $O(E^2)$ and $O(E^4)$. The two have several complementary features, as well as some limitations, and these are described by the use of examples, which include chiral sum rules related to the Weinberg sum rules, form factors, and a more complicated reaction, $\gamma\gamma \to \pi\pi$.

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Theoretical predictions are transformations from one set of known data to a new set of data which we want to know. In renormalizable field theories one predicts many observables in terms of the results of a few experiments measuring the fundamental constants e and m. In chiral perturbation theory, measurements of a few low energy constants (F_π, L_i), which compactly summarize the effects of QCD, allow calculations of other processes. Dispersion relations make predictions in the same sense, transforming measurable data in some process into predictions of the physics elsewhere.

It is well known that in QCD one can make rigorous predictions at high energy in a perturbative expansion in α_s, although one also needs structure functions which are not perturbatively calculable and which must be measured. It is becoming better known that rigorous results can also be obtained at very low energy using chiral perturbation theory, with calculations organized in an expansion in the energy. However the intermediate energy region is the most difficult. This is traditionally the realm of models, such as the quark model, Skyrme model etc. which, while capturing some of the physics, suffer from a lack of control. Dispersion relations can sometimes be used to replace this model dependence by experimental data. In principle, dispersion relations form a rigorous technique for this intermediate energy region. In practice, our knowledge of the input to dispersion relations is often somewhat incomplete, so that some model dependence may enter, but it can often be controlled or bounded to an acceptable level. Done properly, dispersive techniques will always enhance the reliability/range of chiral calculations.

In this talk, I briefly describe chiral perturbation theory and dispersion relations separately. Although they have different starting points, it becomes clear that the contents of chiral loops and of the dispersive integrals are basically similar. The remaining features of the two methods are complementary and we can match the two descriptions in ways that are mutually beneficial. I describe how this matching occurs at different orders in the energy expansion, and give examples of what is gained from doing so.

1 Chiral Perturbation Theory

Chiral symmetry provides relations between amplitudes with different numbers of zero energy pions.[1,2,3] Corrections to this limit can be given in an expansion in the energy. There exist various reduced matrix elements which

are not predicted by the symmetry and which therefore must be measured, at least until reliable methods succeed in predicting them from QCD. According to the power counting theorem of Weinberg[3], at order E^2 in the energy expansion one needs to consider tree level processes and the only incalculable parameters are the pion mass, m_π, and its decay constant, F_π. At order E^4, one has a modest set of low energy constants[2], $L_i, i = 1 \ldots 10$, and in addition one must include one loop diagrams. At order E^6, one calculates to two loops and has a formidable array of low energy constants. It is unlikely that order E^6 calculations will be practical without the use of models to estimate these parameters.

An example, which will also be useful later, is the pion electromagnetic form factor. At lowest order $O(E^2)$, one predicts simply that $f_\pi(q^2) = 1$, while the tree level contribution at $O(E^4)$ involves the low energy constant L_9 with a q^2 dependence, plus constant terms i.e.,

$$f_\pi^{(tree)}(q^2) = 1 + \frac{2L_9}{F_\pi^2}q^2 + \frac{8m_\pi^2}{F_\pi^2}(2L_4 + L_5) \tag{1}$$

Of the loop diagrams, Fig. 1b, c, that of Fig. 1b has no q^2 dependence, contributing only a constant

$$\Delta f_\pi^{(1b)}(q^2) = \frac{-5m_\pi^2}{48\pi^2 F_\pi^2}\left\{\frac{2}{d-4} + \gamma - 1 - ln4\pi + ln\frac{m_\pi^2}{\mu^2}\right\} \tag{2}$$

Since we know that the pion form factor is absolutely normalized to unity at $q^2 = 0$, we know that this constant must be canceled by the wavefunction renormalization constant along with the diagram in Fig. 1c, but the latter also contains physics which is more interesting. Fig. 1c also contains important dynamical information of the propgagation of the two pion state, including the imaginary part of the amplitude due to on-shell intermediate states, and the result involves a nontrivial function of q^2,

$$\Delta f_\pi^{(1c)}(q^2) = \frac{1}{16\pi^2 F_\pi^2}\left\{\left(m_\pi^2 - \frac{1}{6}q^2\right)\left[\frac{2}{d-4} + \gamma - 1 - ln4\pi + ln\frac{m_\pi^2}{\mu^2}\right] + \frac{1}{6}\left(q^2 - 4m_\pi^2\right)H(q^2) - \frac{1}{18}q^2\right\} \tag{3}$$

with

$$
\begin{aligned}
H(q^2) &= 2+\beta\left[ln\left(\frac{1-\beta}{1+\beta}\right)+i\pi\theta(q^2-4m_\pi^2)\right] \qquad (4)\\
\beta &= \sqrt{1-\frac{4m_\pi^2}{q^2}}
\end{aligned}
$$

Multiplying by the wavefunction renormalization constant

$$
Z_\pi = 1-\frac{8m_\pi^2}{F_\pi^2}(2L_4+L_5)+\frac{m_\pi^2}{24\pi^2F_\pi^2}\left\{\frac{2}{d-4}+\gamma-1-ln4\pi+ln\frac{m_\pi^2}{\mu^2}\right\} \qquad (5)
$$

and defining the renormalized value of L_9

$$
L_9^r = L_9-\frac{1}{192\pi^2}\left[\frac{2}{d-4}-ln4\pi+\gamma-1\right] \qquad (6)
$$

we get the final result

$$
f_\pi(q^2) = 1+\frac{2L_9^r}{F_\pi^2}q^2+\frac{1}{96\pi^2F_\pi^2}\left[(q^2-4m_\pi^2)H(q^2)-q^2ln\frac{m_\pi^2}{\mu^2}-\frac{q^2}{3}\right] \qquad (7)
$$

What then is the content of chiral loops? It is easy to state what are *not* important features of the loops. First of all the high energy behavior of the loop is not relevant, because we are using forms of the vertices which are valid only at low energy. This ensures that the high energy portions of the diagrams are not correct. In a similar vein, the divergences are not the important physics since they are part of the high energy structure and do not correspond to the divergences of QCD diagrams. Fortunately, all high energy effects can be absorbed in a shift in the low energy constants. This is true because the high energy portions must obey the symmetries of the theory and must be local effects when external particles carry only low energy. They are thus equivalent to a local term in an effective Lagrangian. Likewise, diagrams such as Fig. 1b do not have interesting physics because they just are universal constants. In a Feynman diagram approach, these are needed to enforce symmetry properties. But they play no dynamical role and if we had other ways to enforce symmetry constraints, as we will in a dispersive approach, they would not be needed.

The important loop physics comes from low energy intermediate states in diagrams such as Fig. 1c. This represents long range propagation and cannot be mimicked by a shift of a parameter in a local chiral Lagrangian. Note the imaginary part which arises from this amplitude. This represents the effects of unitarity coming from physical intermediate states. Unitarity is satisfied order by order in the energy expansion. At one loop one uses the lowest order couplings in the vertex and propagation without any rescattering in the intermediate state. Higher order loop diagrams would allow for modification of the vertices and for rescattering in the intermediate state.

The outputs of chiral perturbation theory are relations between amplitudes, order by order in the expansion in E, m_q. At any given order, these relations form low energy theorems of QCD. However, in applying the method in phenomenology, we often push it to regions where it is less accurate than desirable. In scattering amplitudes it is always tempting to use chiral perturbation theory to describe reactions at higher energies until eventually the result at a given order must break down. Similarly, in some calculations with kaons, the first known corrections are large enough that we would like to know yet higher order corrections. Despite the beauty of the method, in phenomenological applications the two main limitations are the fact that amplitudes are known only to a limited order in the energy expansion and the proliferation of unknown constants at order E^6 and higher.

2 Dispersion Relations

Scattering amplitudes and vertex functions will in general contain both real and imaginary parts.[4,5] The imaginary portions are due to the propagation of on-shell intermediate states. Causality implies certain properties of the analytic structure of the amplitudes that it allows us to relate the real and imaginary parts. Such dispersion relations have the general form

$$Ref(s) = \frac{1}{\pi} P \int_0^\infty \frac{ds'}{s'-s} Imf(s') \tag{8}$$

With the identity

$$\frac{1}{x - x_0 - i\epsilon} = P\frac{1}{x - x_0} + i\pi\delta(x - x_0) \tag{9}$$

one can write the full amplitude as an integral over its imaginary part,

$$f(s) = \frac{1}{\pi}\int_0^\infty ds' \frac{Imf(s')}{s' - s - i\epsilon} \tag{10}$$

Notice that the dispersive integral involves all s'. In order to know $f(s)$ at small s, we need to know $Imf(s')$ also at large s'. We will see that subtractions can lessen the dependence on large s', but the integral still runs over all s'. We in general need to know the properties of on shell intermediate states.

All Feynman diagrams share the required analytic structure and can be rewritten as dispersion relations, perhaps with subtractions. Therefore the content of chiral loops can equally well be specified as a particular choice for $Imf(s')$ in a dispersion integral. When it is phrased this way, it is clear that the content of chiral loop diagrams such as Fig. 1c and the content of a dispersive integral are similar. The chiral calculations uses a predicted approximation to $Imf(s')$, while a properly performed dispersion integral uses the real world data for $Imf(s')$. We will also see that the chiral parameters (L_i) play a similar role to the subtraction constants in dispersion relations.

3 Example: The Weinberg sum rules and some relatives

The simplest amplitudes are two point functions, and within QCD the simplest of these are the particular combination of vector and axial vector currents.

$$\pi_V^{\mu\nu}(q^2) - \pi_A^{\mu\nu}(q^2) \equiv i\int d^4x e^{iq\cdot x}\langle 0 \mid T\left[V^\mu(x)V^\nu(0) - A^\mu(x)A^\nu(0)\right] \mid 0\rangle \tag{11}$$

This combination is analytic in the complex q^2 plane, except for a pole at $q^2 = m_\pi^2$ and a cut for $q^2 > 4m_\pi^2$. The vector current is conserved. The axial current is conserved in the $m_q \to 0$ limit, but with a Goldstone boson. If we define scalar function by

$$\begin{aligned} \pi_V^{\mu\nu}(q^2) &= (q^\mu q^\nu - g^{\mu\nu}q^2)\pi_V(q^2) \\ \pi_A^{\mu\nu}(q^2) &= (q^\mu q^\nu - g^{\mu\nu}q^2)\pi_A(q^2) - q^\mu q^\nu \pi_A^{(0)}(q^2) \end{aligned} \tag{12}$$

we can prove the dispersion relations

$$\pi_V(q^2) - \pi_A(q^2) = \frac{F_\pi^2}{q^2} + \int_{4m_\pi^2}^{\infty} ds' \frac{\rho_V(s') - \rho_A(s')}{s' - q^2 - i\epsilon} \tag{13}$$

with the imaginary parts conventionally named via

$$\rho_{V/A}(s) = \frac{1}{\pi} Im \pi_{V/A}(s) \tag{14}$$

What is known theoretically about these amplitudes? At low energy, chiral perturbation theory predicts the form[2]

$$\begin{aligned}
\pi_V^{\mu\nu}(q^2) - \pi_A^{\mu\nu}(q) &= \left[\frac{q_\mu q_\nu}{q^2 - m_\pi^2 + i\epsilon} - g_{\mu\nu}\right] F_\pi^2 \\
&+ (q_\mu q_\nu - g_{\mu\nu} q^2) \left[\frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{q^2}\right) H(q^2) - 4L_{10}^r \right. \\
&- \left. \frac{1}{48\pi^2}\left(ln\frac{m_\pi^2}{\mu^2} + \frac{1}{3}\right)\right] \\
\rho_V(s) &= \frac{1}{48\pi^2}\left[1 - \frac{4m_\pi^2}{s}\right]^{\frac{3}{2}} \theta(s - 4m_\pi^2) + O(s) \\
\rho_A(s) &= \frac{s}{96(4\pi F_\pi)^2} + O(s^2)
\end{aligned} \tag{15}$$

Here L_{10}^r is a low energy constant measured in radiative pion decay, $\pi \to e\nu\gamma$.

At high energy, pertrubative QCD may analyse the two point function. In the chiral limit, $m_q = 0$, which will be used for the rest of this section, the operator product expansion can be used to show that the difference $\pi_V - \pi_A$ falls as $\frac{1}{q^6}$ and $\rho_V(s) - \rho_A(s) \sim \frac{1}{s^3}$. In terms of four quark operators, which are here evaluated in the vacuum saturation approximation[6], one has

$$\begin{aligned}
\pi_V(q^2) - \pi_A(q^2) = \frac{32\pi}{9} \frac{\langle\sqrt{\alpha_s}\bar{q}q\rangle_0^2}{q^6} \left\{1 + \frac{\alpha_s(q^2)}{4\pi}\left[\frac{247}{12} + ln\frac{\mu^2}{-q^2}\right]\right\} \\
\rho_V(s) \to \rho_A(s) \to \frac{1}{8\pi^2}\left[1 + \frac{\alpha_s(s)}{\pi}\right] \ , \ s \to \infty \\
\rho_V(s) - \rho_A(s) \sim \frac{8}{9} \frac{\alpha_s \langle\sqrt{\alpha_s}\bar{q}q\rangle_\infty^2}{s^3}
\end{aligned} \tag{16}$$

We see that $\pi_V - \pi_A$ and $\rho_V - \rho_A$ are very well behaved at large q^2, s.

We can combine up this information to get a set of sum rules. The requirement that, as $q^2 \to \infty$, there is no $\frac{1}{q^2}$ term in the dispersion relation Eq.13 , requires

$$F_\pi^2 = \int_0^\infty ds(\rho_V(s) - \rho_A(s)) \tag{17}$$

while the absense of $\frac{1}{q^4}$ implies

$$0 = \int_0^\infty dss(\rho_V(s) - \rho_A(s)) \tag{18}$$

These are the Weinberg sum rules[7], the second of which is only true in the $m_q \to 0$ limit. At low energy, expansion of the dispersion integral and chiral results in powers of q^2 imply[8,2]

$$-4\bar{L}_{10} = \int_{4m_\pi^2}^\infty \frac{ds}{s}(\rho_V(s) - \rho_A(s)) \tag{19}$$

with

$$\begin{aligned} \bar{L}_{10} &= L_{10}^r(\mu) + \frac{1}{192\pi^2}\left[ln\frac{m_\pi^2}{\mu^2} + 1\right] \\ &= (-0.7 \pm 0.03) \times 10^{-2}\ (Expt : \pi \to e\nu\gamma) \end{aligned} \tag{20}$$

Here I have given the sum rule for finite m_π^2 since there is a behavior proportional to lnm_π^2 at the low energy end of the integral. These sum rules illustrate one of the uses of chiral dispersion relations, which is the prediction/calculation of low energy constants (here F_π and L_{10}).

Another use of chiral dispersion relations is in extending the reach of calculations and even opening up the possibility of entirely new types of calculations. Consider the Compton amplitude $\gamma\pi \to \gamma\pi$. In the soft pion limit, chiral symmetry relates this to the vacuum polarization tensors

$$\begin{aligned} &\lim_{p\to 0}\langle\pi^+(p) \mid T(V^\mu(x)V^\nu(0)) \mid Ti^+(p)\rangle \\ &= -\frac{1}{F_\pi^2}\langle 0 \mid T(V^\mu(x)V^\nu(0) - A^\mu(x)A^\nu(0)) \mid 0\rangle \\ &= -\frac{1}{F_\pi^2}\left[\pi_V^{\mu\nu}(x) - \pi_A^{\mu\nu}(x)\right] \end{aligned} \tag{21}$$

If one takes the Compton amplitude and ties together the two electromagnetic currents with a photon propagator, one obtains the pion electromagnetic mass shift.[9] Clearly the chiral representation, Eq. 15, would be inadequate to calculate this, as the photon loop integral goes over all values of q^2. However, after some algebra plus the application of the Weinberg sum rules, the dispersive representation allows one to write this as

$$m^2_{\pi^+} - m^2_{\pi^0} = -\frac{3e^2}{16\pi^2 F^2_\pi}\int_0^\infty dss ln s\left[\rho_V(s) - \rho_A(s)\right] \tag{22}$$

which is an exact relation in the chiral limit. Note that here chiral symmetry was used to relate different amplitudes in Eq. 23 and to provide low energy constraints , as in the Weinberg sum rules, while dispersion relations were needed to provide a predictive framework for the intermediate energy region.

In a similar way, one can calculate reliably a new weak nonleptonic matrix element.[10] Consider the hypothetical weak Hamiltonian

$$H_V = \frac{g_2^2}{8}\int d^4x i D^{\mu\nu}_F(x, M_w) T\left(\bar{d}(x)\gamma_\mu u(x)\bar{u}(0)\gamma_\nu S(0)\right) \tag{23}$$

Up to some KM factors, this would be the usual weak Hamiltonian if the vector currents were replaced by $\gamma_\mu(1+\gamma_5)$. In the chiral limit, we have another chiral sum rule

$$\langle \pi^- \mid H_V \mid K^-\rangle = \frac{3iG_F}{32\pi^2\sqrt{2}F^2_\pi}A \tag{24}$$

with

$$A = M^2_w\int_0^\infty ds\frac{s^2 ln(s/M^2_w)}{s - M^2_w + i\epsilon}\left[\rho_V(s) - \rho_A(s)\right] \tag{25}$$

which is exact in the chiral limit.

Gene Golowich and I have recently provided a phenomenological analysis of these sum rules.[11] The physics of the spectral functions $\rho_{V,A}$ is basically simple. At intermediate energies they are measured in τ decay and e^+e^- annihilation, and the largest features are the ρ and a_1 resonances, with very much smaller $4\pi, 5\pi$ etc. contributions. At low energy this can be merged smoothly to chiral predictions and at high energy $\rho_V - \rho_A$ vanishes rapidly and we matched the data to QCD around $s = 5GeV^2$. There are some experimental uncertainties, but these can in principle be reduced in the future.

The L_{10} sum rule works well with very little uncertainty as it is sensitive to the lowest energy contributions. The Weinberg sum rules and that for Δm_π^2 work within the experimental uncertainties. We have proceeded by imposing them exactly on our $\rho_V - \rho_A$, which requires only minor adjustments within the allowed error bars. That this is possible is a nontrivial test of the theoretical framework. Finally the weak matrix element is predicted ($A = -0.062 \pm 0.017 GeV^2$). This can perhaps be compared with future lattice calculations.

4 Subtractions

Given a dispersion relation, one may also write a "subtracted" relation for $(f(q^2) - f(0))/q^2$, i.e.,

$$\frac{f(q^2) - f(0)}{q^2} = \frac{1}{\pi} \int \frac{ds'}{s' - q^2 - i\epsilon} Im \left[\frac{f(s') - f(0)}{s'} \right] \tag{26}$$

which, since $Imf(0) = 0$ is equivalent to

$$f(q^2) = f(0) + \frac{q^2}{\pi} \int \frac{ds'}{s'} \frac{Imf(s')}{s' - q^2 - i\epsilon} \tag{27}$$

This may be needed if $f(z) \neq 0$ at $\mid z \mid \rightarrow \infty$, as a good behavior at infinity is required for the derivation of the dispersion relation. However, even if subtractions are not required, it may still be desirable to perform them. Generally $Imf(s)$ is not well known at high energy. The subtracted dispersion integral weights lower energies more heavily and lessens the influence of the high energy region. The previous ignorance of the high energy effects of $Imf(s)$ is reduced to a single number, the subtraction constant. Further subtractions may be performed, with the introduction of further subtraction constants.

The pion form factor obeys dispersion relations. An unsubtracted form is

$$f_\pi(q^2) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{Imf_\pi(s')}{s' - q^2 - i\epsilon} \tag{28}$$

while with one subtraction the form is

$$f_\pi(q^2) = 1 + \frac{q^2}{\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{Imf_\pi(s')}{s'-q^2-i\epsilon} \tag{29}$$

Here the subtraction constant has been fixed to unity by the normalization of the form factor. A twice subtracted form is

$$f_\pi(q^2) = 1 + cq^2 + \frac{q^4}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{Imf_\pi(s')}{s'-q^2-i\epsilon} \tag{30}$$

where c is presently unknown.

5 Matching Conditions

In dispersion relations involving subtraction constants we need a precise identification of them. Chiral perturbation theory provides an extensive machinery for the analysis of the low energy behavior and can provide these constants.[12] The key is to reformulate chiral calculations as dispersion relations, order by order. As mentioned previously this is always possible because the Feynman diagrams themselves satisfy dispersion relations. An important point is that the matching is different at order E^2 [13] and at order E^4 [14,15].

At order E^2 one needs to reproduce only the tree level chiral results, which do not involve imaginary parts. Thus we only need to ensure that the normalization at low energy is correct. The dispersion integral will then produce new effects at order E^4 which are equivalent to the prediction of the low energy constants at order E^4, i.e., of the L_i. This procedure will be more sensitive to high energy effects because one will be using a dispersion integral with at most one subtraction.

At order E^4 one knows more about the low energy structure so one can use a dispersion relation with an extra subtraction. The low energy constants L_i are no longer predicted, but are inputs to fix the subtraction constants [The dispersion integral then produces new effects at order E^6 and higher]. To match at this order one must reproduce the one loop chiral calculation. Therefore the inputs to the dispersive integral must involve the lowest order vertices, and will only have free propagations of the intermediate state, i.e., the same inputs that go into the Feynman diagram calculation.

As an example, let us first consider the pion form factor with matching at order E^4.[14] The one loop diagram, Fig. 1c, involves the $\pi\pi I = 1$ scattering amplitude, and the tree level $\pi\pi \to \gamma$ vertex, so that

$$2(p_1-p_2)_\mu Imf_\pi(s) = \int \frac{d^3p_1' d^3p_2'}{(2\pi)^6 2E_1' 2E_2'}(2\pi)^4\delta^4(s-p_1'-p_2')\langle\pi\pi \mid T \mid \pi\pi\rangle\langle\pi\pi \mid J_\mu \mid 0\rangle \tag{31}$$

or

$$Imf_\pi(s) = \frac{1}{96\pi F_\pi^2}\frac{(s-4m_\pi^2)^{\frac{3}{2}}}{\sqrt{s}}\theta(s-4m_\pi^2) \tag{32}$$

We use the twice subtracted form, Eq. 30, and the dispersion integral can be exactly done using

$$\int_{4m^2}^{\infty}\frac{ds}{s^2}\left(1-\frac{4m^2}{s}\right)^{\frac{1}{2}}\left(\frac{a+bs}{s-q^2-i\epsilon}\right) = \frac{(a+bq^2)}{q^4}H(q^2) - \frac{a}{6m^2q^2} \tag{33}$$

to give

$$f_\pi(q^2) = 1 + cq^2 + \frac{1}{96\pi^2F_\pi^2}\left[(q^2-4m_\pi^2)H(q^2) + \frac{2}{3}q^2\right] \tag{34}$$

Comparing this with the chiral calculation, Eq. 7, leads to the identification of the subtraction constant

$$c = \frac{2L_9^{(r)}}{F_\pi^2} - \frac{1}{96\pi^2F_\pi^2}\left(ln\frac{m_\pi^2}{\mu^2}+1\right) \tag{35}$$

If we now return to the full dispersive integral, we must be sure that the input $Imf_\pi(s)$ agrees with the lowest order chiral result at low energies, which of course it must in any case if chiral symmetry is a valid description at low energy. The use of the experimental $Imf_\pi(s)$, thus constrained, then generates the full $f_\pi(q^2)$ at all q^2. In principle, the only inaccuracy in this calculation is that we have given the subtraction constant c by an expression which is exact only to order E^4. There can be corrections to this by extra factors of m_π^2 or $m_\pi^2 lnm_\pi^2$.

Let us also briefly consider the same quantity matched at $O(E^2)$, using Eq. 29. Now the only matching is the simple constraint $f_\pi(0) = 1$, and the effect of the dispersive integral starts at q^2. This leads to a prediction of the low energy constant

$$2L_9^r + \frac{1}{96\pi^2}\left(ln\frac{m_\pi^2}{\mu^2} + 1\right) = F_\pi^2 \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} Imf_\pi(s') \tag{36}$$

Note that the lowest order form of $Imf_\pi(s)$ cannot be inserted in the once subtracted dispersion integral, as the result diverges. The lowest order form for $Imf_\pi(s)$ is not valid at high energies, but the twice subtracted integral used above was not sensitive to this. The use of the real data for $Imf_\pi(s')$ leads to a succesful prediction of L_9^r in terms of the mass of the rho meson.

From these examples, we can see clearly the dynamical content of dispersion relations. If minimally subtracted (i.e., just barely convergent) this includes the effects of low, moderate and high energy intermediate states. If oversubtracted (i.e., more than is required by the high energy behavior), high energy effects are damped and we retain the effects of low and moderate energy propagation, and can remain consistent with the chiral constraints while extending the calculation to higher q^2.

Thus in the best of all worlds (full data on $Imf(s)$, many related reactions) the two techniques form a powerful combination which allows rigorous results at all energies. Chiral perturbation theory provides the subtraction constants from symmetry relations and dispersion relations allows the extrapolation to higher energy.

In real world phenomenology, we often have incomplete information. In addition, we may want/need to predict Imf also.

6 The Elastic Approximation and the Omnes Problem

Consider some two particle amplitude $f(s)$ of a given isospin and angular momentum which is analytic in complex s plane except for a cut above two particle threshold $s_0 = 4m^2$. The inelastic thresholds are somewhat higher, for example $s_{inel} = 16m^2$. In the elastic region, Watson's theorem tells us that the phase of the amplitude is that of the corresponding two particle

scattering amplitude

$$f(s) = e^{i\delta(s)} \mid f(s) \mid \tag{37}$$

In practice inelasticities do not play a significant role in low energy pion physics up to 1 GeV ($K\bar{K}$ threshold), and one often assumes an approximation of keeping only the elastic channel. While probably reasonable, it is important to realize that the elastic approximation relies on more than just Watson's theorem and produces more than just the phase of the amplitude.

The Omnes problem[16] is the mathematical exercise of finding functions which are analytic except for a cut $4m^2 < s < \infty$, which are real when s is real and less than $4m^2$ and for which $f(s)e^{-i\delta(s)}$ is real when s is real and greater than $4m^2$. The solution is given by

$$\begin{aligned} f(s) &= P(s)D^{-1}(s) \\ D^{-1}(s) &\equiv exp\left\{\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{dt}{t}\frac{\delta(t)}{t-s-i\epsilon}\right\} \end{aligned} \tag{38}$$

as long as

$$\lim_{s\to\infty}\delta(s) = finite\ ;\ \lim_{s\to\infty}\frac{\mid f(s) \mid}{s} \to 0 \tag{39}$$

In the above $P(s)$ is a polynomical in s, and $D^{-1}(s)$ is called the Omnes function.

Note that this is not exactly the right problem for QCD. The assumption that $f(s)e^{-i\delta(s)}$ is real above the cut implies that the reaction is elastic at *all* energies. Once inelastic channels open up, the quantity $f(s)e^{-i\delta(s)}$ rapidly deviates from being real. In QCD, once one is above 1 GeV, the inelastic channels open rapidly and become quite numerous, leading to perturbative QCD behavior at precociously low energies. It is not known how to provide a general solution to the QCD type problem (although the form of the solution to the two channel problem is also known), nor is it known how much of an effect the multiple inelasticies of QCD have on the Omnes function.

7 Example: Matching at order E^2

One of the earliest examples of the utility of merging chiral perturbation theory and dispersion relations came in the analysis of the possible decay of a light Higgs to two pions.[13] This is no longer of phenomenological interest, but the technique developed illustrates the basic methodology. Among other couplings, one is faced with the matrix of the energy momentum tensor,

$$\langle \pi^i(p)\pi^j(p')) \mid \theta^\mu_\mu \mid 0\rangle = \theta_\pi(s)\delta^{ij} \tag{40}$$

At lowest order, chiral perturbation theory tells us that

$$\theta_\pi(s) = s + 2m_\pi^2 \tag{41}$$

while at order E^4 the form is

$$\theta_\pi(s) = (s + 2m_\pi^2)\left\{1 + \phi(s)\right\} + b_\theta s^2 + O(m_\pi^4, sm_\pi^2, E^6) \tag{42}$$

where b_θ is a combination of the low energy constants L_{11}, L_{12}, L_{13} that occur when one analyses the energy momentum tensor[17] and $\phi(s)$ is a known loop function. The trouble is that we have no phenomenology which measures b_θ. However, $\theta_\pi(s)$ can be shown to satisfy a dispersion relation, and the elastic region only involves $\pi\pi$ scattering in the $I = 0, J = 0$ channel. Therefore, in the elastic approximation, we can reproduce both the right chiral properties and satisfy the Omnes solution by choosing

$$\theta_\pi(s) = (2m_\pi^2 + s)D^{-1}(s) \tag{43}$$

[In practice, a two channel solution was found, involving $K\bar{K}$ states above 1 GeV. However, for our example, let us neglect $K\bar{K}$.] The Omnes function was constructed from $\pi\pi$ data by Gasser.[18]

The output of this representation is a prediction for $\theta_\pi(s)$ at higher energies than is possible with chiral perturbation theory alone. One of the by products is a prediction of the low energy constant.

$$b_\theta = 2.7GeV^{-2} \tag{44}$$

In this particular channel, the effects of rescattering are significant. For example at $\sqrt{s} = 0.5GeV$, the lowest order formula gives

$$\theta_\pi(s) = 0.29 \tag{45}$$

while at one loop, $O(E^4)$ with the above value of b_θ one finds

$$\theta_\pi(s) = 0.46 + 0.13_i \tag{46}$$

and the full dispersive amplitude is

$$\theta_\pi(s) = 0.40 + 0.31_i \tag{47}$$

This large effect is typical of the $I = 0, J = 0$ $\pi\pi$ amplitude, which gets large very quickly, so much so that the lowest order chiral prediction for it violates unitarity around 600 MeV. When the $I = 0, J = 0$ $\pi\pi$ scattering channel is important in a calculation, a dispersive treatment could be useful. It is also typical that one can obtain good agreement for the magnitude $\mid \theta_\pi(s) \mid$ with a suitably chosen low energy constant at $O(E^4)$, but that the imaginary part will be too small at one loop when compared to the full answer. This is because the one loop imaginary part corresponds to using the lowest order amplitudes in the unitarity relation. This is not a big problem, because it could be corrected by hand as the second order amplitudes are known and we could use these to determine the second order imaginary part. However, the dispersive treatment does this automatically, as well as including yet higher orders.

8 Example: Matching at Order E^4

The reaction $\gamma\gamma \to \pi^+\pi^-$ and $\gamma\gamma \to \pi^0\pi^0$ are of interest in the development of chiral theory because $\gamma\gamma \to \pi^0\pi^0$ first arises as a pure loop effect as there are not tree level contributions at $O(E^2)$ or $O(E^4)$. For these reactions, we have both a one-loop [19]and two loop [20]chiral analysis as well as dispersive treatments[21,15] and experimental data. This makes these reactions excellent illustrations of chiral techniques and of the ties with dispersion relations.

The $\gamma\gamma \to \pi\pi$ matrix elements can be decomposed into isospin amplitudes

$$\begin{aligned} f^{+-}(s) &= \frac{1}{3}[2f_0(s) + f_2(s)] \\ f^{\infty}(s) &= \frac{2}{3}[f_0(s) - f_2(s)] \end{aligned} \tag{48}$$

The dominant partial waves at low energy are the S waves and these are predicted in a one loop chiral analysis to be

$$\begin{aligned} f_I^{chiral}(s) &= \frac{1-\beta^2}{2\beta} ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{(1-\beta^2)}{4\pi} t_I^{CA}(s) ln^2 \frac{\beta+1}{\beta-1} \\ &\quad -\frac{1}{\pi} t_I^{CA}(s) + \frac{2}{F_\pi^2}\left(L_9^r + L_{10}^r\right) s \end{aligned} \tag{49}$$

where

$$\beta = \sqrt{1 - \frac{4m_\pi^2}{s}} \tag{50}$$

and $t_I^{CA}(s)$ are the lowest order $\pi\pi$ scattering amplitudes

$$t_0^{CA} = \frac{2s - m_\pi}{32\pi F_\pi^2} \; ; t_2^{CA} = -\frac{(s - sm_\pi^2)}{32\pi F_\pi^2} \tag{51}$$

The dispersion relation has been derived by Morgan and Pennington[21], in terms of an amplitude $p_I(s)$ which has the same left-hand singularity structure as $f_I(s)$ but which is real for $s > 0$. Then $[f_I(s) - p_I(s)]D_I(s)$ satisfyies a twice subtracted dispersion relation and we have

$$f_I(s) = D_I^{-1}(s)\left[(C_I + d_I s) + p_I(s)D_I(s) - \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{p_I(s') ImD_I(s')}{s' - s - i\epsilon}\right] \tag{52}$$

with two subtaction constants per channel c_I, d_I. As a prelude to the matching we note that Low's theorem requires that $f_I(s)$ be the Born scattering amplitude at low energies. Therefore

$$p_I(s) = f_I^{Born}(s) + O(s) = \frac{1-\beta^2}{2\beta} ln\left(\frac{1+\beta}{1-\beta}\right) + O(s) \tag{53}$$

This is the $O(E^2)$ result. To proceed to order E^4 we note that the leading piece of $ImD_I(s)$ is also known, i.e.,

$$ImD_I(s) = -\beta t_I^{CA}(s) \tag{54}$$

as this is the lowest order $\pi\pi$ scattering amplitude. Using this, the dispersive integral can be done exactly, leading to

$$\begin{aligned} f_I(s) &= D^{-1}(s)\left[c_I + s\left(d_I - \frac{t_I^{CA}(0)}{12\pi m_\pi^2}\right) + D_I(s)\frac{1-\beta^2}{2\beta} ln\left(\frac{1+\beta}{1-\beta}\right)\right. \quad (55) \\ &- \left.\frac{1}{4\pi}(1-\beta^2)t_I^{CA}(s) ln^2\left(\frac{\beta+1}{\beta-1}\right)\right] + \ldots \end{aligned}$$

A comparison of this with the $O(E^4)$ chiral results then indicates that this procedure has reproduced all of the one loop results, as long as we choose the subtraction constants as[15]

$$c_I = 0\ ; d_I = \frac{2}{F_\pi^2}\left(L_9^r + L_{10}^r\right) + \frac{t_I^{CA}(0)}{12\pi m_\pi^2} \quad (56)$$

Again we see that the dynamical content of the one loop chiral calculation is also contained in the dispersive treatment when the imaginary part is taken to be the lowest order scattering amplitude. However, chiral symmetry also predicts the subtraction constants, which in this case are known from measurements in radiative pion decay.

Having identified the subtraction constants one can add the ingredients to complete the calculation. The most important at threshold is the use of the real world $D_I^{-1}(s)$[18]. This change alone produces a significant effect in the amplitude even near threshold in the neutral case. The second step is a better determination of $p_I(s)$ which includes the $O(E^4)$ chiral corrections to it as well as the $\rho, \omega, A1$ poles which are known (from $\rho \to \pi\gamma$ etc. data) to occur in the Compton amplitude. Figure 2 shows the data for the reaction $\gamma\gamma \to \pi^0\pi^0$ along with the one-loop chiral prediction (dashed line) and the modification obtained by the dispersive treatment (solid line). The one-loop chiral result is of the right rough size, its slope is low at threshold and it grows unphysically at high energy. Near threshold the difference in the two calculations comes almost exclusivly from the rescattering corrections generated through the dispersion relation. The change is sizeable even at low energy, since the rescattering in in the $I = 0, J = 0$ channel. . The Omnes function alone has brought the threshold region into better agreement with the data. It has also tamed the high energy growth. The final result (with

no free parameters) matches the data very well, and also gives the charged channel correctly.

Belucci, Gasser and Sainio[20] have performed the enormously difficult two loop calculation. [In fact, technically they employ dispersive methods to do portions of this.] At two loop order, new low energy constants appear, which are not measured in any other process. Therefore the authors have to step outside of pure chiral perturbation theory in order to model these constants, using vector meson dominence. Much like the dispersive work described above, these constants play little role in the threshold region, but are important for the shape of the amplitude for moderate energy. It is very interesting that their results look very similar to the dispersive treatment described above.

Both of these methods have potential limitations. In principle, the only limitation of the dispersive treatment is the fact that it can miss $O(E^6)$ terms in the subtraction constants c_I, d_I. These would be corrections to results given above by factors of m_π^2 or $m_\pi^2 ln m_\pi^2$. In practice we also need to model the higher order terms in $p_I(s)$. As for the two loop chiral result, its only limitation in principle is the fact that it misses higher order dependnce in the energy. By construction it is valid to order E^6, but does not contain higher order s dependence, and so would be expected to fall apart soon after the E^6 dependence became important. In practice, this approach also needs to do some modeling in order to estimate the unknown low energy constants. The fact that the results agree so well with each other and with the data indicates that these limitations are not very important at these energies. Both capture the important physics, and do so in a reasonably controlled fashion. There is of course a significant practical advantage to the dispersive approach–it is far easier!

9 Summary

We have seen how dispersion relations can add power to chiral perturbation theory. At its best it uses more physics input. It can match all chiral effects to whichever order that they are known, and can be used to replace the modelling of unknown physics by using data instead of models. However there are some limitations, coming both from incomplete data and from the fact that we can only determine the subtraction constants to a given order

in the energy expansion.

The technology for combining these techniques is now developed. This involves first knowing the chiral analysis of the amplitude to a given order in the energy expansion. One also needs a dispersion relation for the amplitude in question. The number of subtractions is determined partially by the high energy behavior of the amplitude, but the use of more subtractions than are required can help in the matching with the chiral result. The matching occurs order by order in the energy expansion. When it can be done, it is preferable to perform the matching at $O(E^4)$ because the resulting dispersive treatment is less sensitive to what happens at high energy since a twice subtracted dispersion relation can be used. Finally, the real world data has to be found to use in the dispersive integral. Often, the use of the elastic approximation is made for this, allowing the use of known $\pi\pi$ scattering data.

The output of these efforts can be several. Most commonly, these techniques are used to extend the range and accuracy of the chiral calculations, by getting around the limitation of the energy expansion. The method can be used to predict unknown chiral coefficients, as was shown for the case of L_{10}. We can use these techniques to remove or reduce the model dependence of some result. Finally, dispersive techniques allow us to perform completely new types of calculations, such as the hadronic matrix elements of Section IV.

There is work going on which is pushing the frontier of what can be done using these techniques. More difficult reactions, such as $K \to \pi\gamma\gamma$ or $K \to \pi e^+ e^-$, require more subtle analyses. Probably more important, the use of dispersive and chiral techniques in hadronic matrix elements can likely be pushed further. Dispersion relations and chiral perturbation theory bring different strengths to their union, and the marriage, although not without an occasional flaw, has been friutful.

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References.

1) J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model*, (Cambridge University Press, Cambridge, 1992).
2) J. Gasser and H. Leutwyler, Nucl. Phys. **B250**, 465(1985).
3) S. Weinberg, Physica **A96**, 327 (1970).

4) R. Kronig, J. Op. Soc. Am. **12**, 547 (1926); H. A. Kramers, Atti Cong. Int. Fisici Como (1927).
5) G. Barton, *Dispersion Techniques in Field Theory*, (Benjamin, NY, 1965).
6) M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979).
L. V. Lanin, V. P. Spiridonov, and K. G. Chetyrkin, Sov. J. Nucl. Phys. **44**, 892 (1986).
7) S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
8) T. Das, V. Mathur, and S. Okubo, Phys. Rev. Lett. **19**, 859 (1967).
9) T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, Phys. Rev. Lett. **18**, 759 (1967).
10) J. F. Donoghue and E. Golowich, Phys. Lett. **315**, 406 (1993).
11) J. F. Donoghue and E. Golowich, Phys. Rev. **D49**, 1513 (1994).
12) As far as I know, the first attempt to use dispersion relations to improve the predictions of chiral perturbation theory occured in Ref. 13. The techniques at order E^4 have been illustrated in Ref 14,15.
13) J. F. Donoghue, J. Gasser and H. Leutwyler, Nucl. Phys. **B343**, 341(1990).
14) T. N. Truong, Phys. Rev. Lett. **61**,2526 (1988).
15) J. F. Donoghue and B. R. Holstein, Phys. Rev. **D48**, 137 (1993).
16) R. Omnes, Nuovo Cim. **8**, 1244(1958).
17) J. F. Donoghue and H. Leutwyler, Z. Phys. **C52**,343(1991).
19) J. Bijnens and F. Cornet, Nucl. Phys. **B296**, 557 (1988).
J. F. Donoghue, B. R. Holstein and Y. C. Lin, Phys. Rev. **D37**, 2423 (1988).
20) S. Bellucci, J. Gasser and M. Sainio, Nucl. Phys. **B423**,801994).
21) D. Morgan and M. R. Pennington, Phys. Lett. **B272**, 134 (1991).